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Spinors?

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1. INTRODUCTION

This is a paper about the consistency of the notion of spinor as it is widely understood and used. Results are easily derived and would be easily foreseen from the beginning, once the blunt question had been properly stated. Nevertheless I think that it is amply motivated by the gaiety with which such consistency is taken for granted in many papers on quantum field theory in curved space--times.

Let (M^n, g) be an oriented pseudo-Riemannian manifold with metric g of index k (the number of negative values in the diagonal form of g). Under certain topological conditions it is possible to have a spinor structure on (M, g), which is a principal bundle $\pi : S \to M$ with group Spin (n, k) together with a principal bundle epimorphism $\Phi : S \to SO(M, g)$, which corresponds to the double covering $\Phi :$ Spin $(n, k) \to SO(n, k)$. Here, $SO(M, g)$ is the principal bundle of oriented g-orthonormal frames. If S is a representation space for Spin (n, k), a spinor (field) is a smooth function $\psi : S \to S$ with the property $\psi \circ R_s = s^{-1} \circ \psi$, where R denotes right action. Thus, spinors can be viewed locally as (perhaps) two-valued equivariant functions on $SO(M, g)$. In this sense, they «live» in $SO(M, g)$, but not, in general, in the whole oriented frame bundle $F_+(M)$.

This seems rather queer from a physical point of view; indeed, one should ask about the required «g-orthonormality» of an apparatus that would be intended to extract some spinorial information about particles. Less metaphysical are the following questions: are spinors compatible with a variational treatment of gravity?, are they compatible with a quantum theory of gravity that would speak about something as the probability of a gravity state? In both cases, one is bound to deal with a family of pseudo-Riemannian matrices, each with its own principal

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bundle $SO(M, g)$, whence the answer is, in principle, in the negative. Spinors are ghettoed to their own principal bundle and forbidden in other places.

This bad feature appears also when they are confronted with the Lie derivative. The best that can be done is the definition given by Y. Kosmann [5] in her thesis, but it is clear that such derivatives do not form a Lie algebra; moreover, if X, Y are vector fields on M and we put $\theta(X, Y) = [LX, LY] - L[X, Y]$, where LX is the Kosmann derivative with respect to X, then θ depends on the 1-jet of X and Y; in other words, θ is not a tensor field that could stand for a sort of curvature. The reason for this is always the same: such derivatives take into account only the «Killing part» of the vector fields; they are unnaturally forced to act as Killing vector fields, that is as fields on $SO(M, g)$, and they rebel against it by showing their non-Killing part in the expression of θ.

So, one is led to ask whether there might be some extension of the definition of spinors that would render them compatible with the different pseudo-Riemannian metrics in a differentiable manifold. In an axiomatic way we ask for a new definition with the following conditions: i) naturality; ii) the new spinors should be as similar as possible to the old ones (i.e. they must be sections of a vector bundle over M with a group containing Spin (n, k), etc.); iii) if a specific pseudo-Riemannian metric on M, g, has been chosen, a new spinor should look formally, from the references in $SO(M, g)$, the same as an old one.

The possible solutions are given by two principal bundles (P, π, M, G), (F, π_F, M, H), the latter being a natural subbundle of $F_+(M)$, together with a double covering $\Psi : G \to H$ and an epimorphism of principal bundles $\Psi : P \to F$ corresponding to Ψ. Leaving out low-dimensional exceptions, four types can appear:

I. $G = G_1$, the double covering (connected if possible) of $H = Gl_+(n; \mathbb{R})$, $F = F_+(M)$.

II. $G = G_2$, the double covering (connected if possible) of $H = Sl(n; \mathbb{R})$, $F = F_+(M)/\mathbb{R}^+$.

III. $G = U(1) \times G_2$, $H = U(1) \times Sl(n; \mathbb{R})$, $F = F_+(M)/Z(r)$, where $Z(r) = \{e^{mr}\}_{m \in \mathbb{Z}}$, $r > 0$.

IV. $G = (\widetilde{U(1)} \times G_2)/\mathbb{Z}_2$, $\widetilde{U(1)}$ the connected double covering of $U(1)$, H and F as in III.

A spinor in the new sense would be a section of a vector bundle associated to P by a representation of G. Of course, since Spin (n, k) is a subgroup of G, the new spinors would appear as usual spinors once given a reduction of $F_+(M)$ to $SO(n, k)$.

Now, by a well known theorem on Lie groups, all finite-dimensional representations of the listed groups G are in fact representations given, via Ψ, by representations of H. Therefore, *there are no finite-dimensional true spinors compatible with the family of pseudo-Riemannian metrics on a smooth oriented manifold*

M. In fact, they are tensor fields in a broad sense.

Thus, for instance, to deal with variations of the metric in a quantum field theory whose momentum-energy tensor contains true spinors is, strictly speaking, meaningless.

A bypass for this disappointment could perhaps be the treatment of fermions recently developed by I.M. Benn and R.W. Tucker [1].

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2. PRELIMINARIES

Let $SO(n, k)$ be the group of linear automorphisms of $\mathbb{R}^n$ that preserve the bilinear form $\langle x, y\rangle = \sum_{i=1}^{n-k} x_i y_i - \sum_{i=n-k+1}^{n} x_i y_i$, which is said to have *index* k. We put $\eta_{ij} = \langle e_i, e_j\rangle$, where the e_i constitute the canonical basis of $\mathbb{R}^n$, and denote by $SO_+(n, k)$ the component of the identity in $SO(n, k)$. $\Phi : \mathrm{Spin}\,(n, k) \to SO(n, k)$ is a double covering with $\ker \Phi = \pm 1$, such that $\mathrm{Spin}_+(n, k) = \Phi^{-1}(SO_+(n, k))$ is connected for $n > 1$ with the exception $n = 2$, $k = 1$. If G is a connected Lie group, we shall put DG to denote the (connected if possible) double covering of G. Let $Gl_+(n; \mathbb{R})$ be the subgroup of $Gl(n; \mathbb{R})$ of matrices with positive determinant, and $\Psi : DGl_+(n; \mathbb{R}) \to Gl_+(n; \mathbb{R})$ the double covering homomorphism. Then $\Psi^{-1}(SO_+(n, k)) \approx \mathrm{Spin}_+(n, k)$ as it is easily proved. Let $\eta : DGl_+(n; \mathbb{R}) \to Gl(S)$ be a representation on the finite-dimensional vector space S. Then η factorizes as $\eta = \tilde{\eta} \circ \Psi$ where $\tilde{\eta}$ is a representation of $Gl_+(n; \mathbb{R})$ on S; the same occurs for $DSl(n; \mathbb{R})$. This is an immediate corollary of well-known results on Lie groups [see 4, XVII, 3.3].

Let M^n, $n \geqslant 1$, be an orientable and oriented differentiable manifold that admits a pseudo-Riemannian metric of index k, g, and $\pi_F : F_+(M) \to M$ be the principal oriented frame bundle. Then $F_+(M)$ has a reduction to $SO(n, k)$ that consists of the oriented g-orthonormal frames $z = (z_i) \in F_+(M)$, that is $g(z_i, z_j) = \eta_{ij}$; we denote this reduction by $\pi_g : SO(M, g) \to M$.

A *spinor structure* over (M, g) is a principal bundle $(S, \pi, M, \mathrm{Spin}\,(n, k))$ together with an epimorphism of principal bundles $\Phi : S \to SO(M, g)$ which corresponds to $\Phi : \mathrm{Spin}\,(n, k) \to SO(n, k)$; that is for each $u \in S$, $s \in \mathrm{Spin}\,(n, k)$ we have $\Phi(R_s u) = R_{\Phi(s)} \Phi(u)$, where R denotes right action. For the sake of clarity we shall write sometimes $u \cdot s$ instead $R_s u$.

We remark that if $U \subset M$ is connected and simply connected, and $\phi : \pi_g^{-1}(U) \to U \times SO(n, k)$ is a trivialization of $SO(M, g)$, then there are exactly two trivializations $\psi : \pi^{-1}(U) \to U \times \mathrm{Spin}\,(n, k)$ of S making commutative the following diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\psi} & U \times \mathrm{Spin}\,(n,k) \\ \Big\downarrow \Phi & & \Big\downarrow (\mathrm{id}, \Phi) \\ \pi_g^{-1}(U) & \xrightarrow{\phi} & U \times SO(n,k). \end{array}$$

Let S be a finite-dimensional vector space on which Spin (n, k) acts through some representation. That action induces a vector bundle $\pi_E : E \to M$ with fiber S, associated to S, whose sections are called *spinors*. Equivalently, one can look at spinors as smooth maps $\xi : S \to S$ such that $\xi \circ R_s = s^{-1} \circ \xi$ for each $s \in$ Spin (n, k).

3. STATEMENT OF THE PROBLEM

Assume that we want to consider variations of g in the sense of variational calculus, or that M stands for space-time and one is looking for a quantization of gravity in such a manner that different states of the metric are to be dealt with. In both cases it is necessary to consider the set of pseudo-Riemannian metrics on M.

Since the definition of spinors demands a previous choice of a metric on M, there is no evidence about the question whether the notion of spinor can be made compatible with different metrics. So, the desire of handling spinors consistently with variational principles that take into account variations on the metric, or the desire of building a quantum theory of gravitation with spinors, leads to the following

PROBLEM. *Find out a notion of spinor compatible with the different pseudo--Riemannian metrics on a differentiable manifold.* ∎

It is obvious that a solution of this problem will be the more suitable as the better it fits the following conditions:

i) naturality;

ii) the new concept must be as similar as possible to the usual notion of a spinor; and this similarity will be the ideal one if

iii) whenever a specific pseudo-Riemannian metric on M, g, has been chosen, the new notion coincides formally with the old.

Let us put gradually in precise terms the preceding conditions. As for ii), we begin by demanding that a (new) spinor on M be a cross section of some vector bundle $\pi_E : E \to M$ with a certain fiber S. We will not specify the vector space S, except that Spin (n, k) should act linearly on S on the left, and that $-1 \in$ Spin (n, k) acts on S as the change of sign, so that the usual prescription that after a 2π rotation a spinor changes sign is retained; this, of course, is also the unique conceivable simple manner of satisfying iii). In brief, we demand that there must be an injection of Spin (n, k) into $Gl(S)$, which makes of Spin (n, k) a closed subgroup of $Gl(S)$, and such that $-1 \in$ Spin (n, k) goes to $-1 \in Gl(S)$.

Let G be the set of pseudo-Riemannian metrics of index k on M. We can identify G with the set of cross sections of the quotient bundle $F_+(M)/SO(n, k)$, denoted by $\Gamma(F_+(M)/SO(n, k))$. To satisfy iii) we demand that to each $g \in G$ there should be associated some reduction of the group of E to Spin (n, k). And if $\mu(g)$ is the principal bundle of the bases of E defined by that reduction, there must be a principal bundle epimorphism $\Phi_g : \mu(g) \to SO(M, g)$ corresponding to Φ : Spin $(n, k) \to SO(n, k)$. Thus, a section of E, when viewed from the points of $\mu(g)$ only, looks exactly as a usual spinor, the only possible difference being the choice of the fiber S.

Let $(P, \pi, M, Gl(S))$ be the principal bundle of bases of the vector bundle E. Since we suppose that Spin (n, k) is a closed subgroup of $Gl(S)$, the reductions of E to the group Spin (n, k) are in one-to-one correspondence with the sections of the bundle $P/$Spin (n, k). Condition iii) demands, therefore, the existence of a map $\mu : \Gamma(F_+(M)/SO(n, k)) \to \Gamma(P/\text{Spin}\,(n, k))$, and also a principal bundle epimorphism $\Phi_g : \mu(g) \to SO(M, g)$ for each $g \in G$, such that $\Phi_g(R_s u) = R_{\Phi(s)} \Phi_g(u)$ for all $s \in$ Spin (n, k), $u \in \mu(g)$.

Now we can elaborate on condition i). The most obvious condition of naturality consists of requiring that μ and Φ_g should be «fibered». That is, μ must proceed from a map, which we keep denoting by μ,

$$\begin{array}{ccc} F_+(M)/SO(n, k) & \xrightarrow{\ \mu\ } & P/\text{Spin}\,(n, k) \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

that makes commutative this diagram and is locally trivial. Local triviality means that we can locally model μ and Φ_g into standard maps.

Hence, we must have a smooth map

$$\mu_0 : Gl_+(n; \mathbb{R})/SO(n, k) \to Gl(S)/\text{Spin}\,(n, k)$$

such that $\mu_0(SO(n, k)) =$ Spin (n, k). This last condition is not strictly necessary but it facilitates computations and does not mean a loss of generality due to the homogeneity of both manifolds. Also, for each class $A \cdot SO(n, k) \in Gl_+(n; \mathbb{R})/SO(n, k)$ we must have a smooth surjective map $\Phi_A : \mu_0(A \cdot SO(n, k)) \to$ $\to A \cdot SO(n, k)$ such that $\Phi_A(a \cdot s) = \Phi_A(a) \cdot \Phi(s)$ for all $a \in \mu_0(A \cdot SO(n, k))$, $s \in$ Spin (n, k).

As for the local modelling of μ and Φ_g, we require the existence of an open covering $\{U_j\}_{j \in J}$ of M (for instance, the U_j being simply connected, see remark at the Introduction), such that for each $j \in J$ and each trivialization (U_j, ϕ_j) of $F_+(M)$, there must exist a trivialization (U_j, ψ_j) of P that makes commutative the following diagram

$$
\begin{array}{ccc}
\pi^{-1}(U_j)/\mathrm{Spin}\,(n,k) & \xrightarrow{\tilde{\psi}_j} & U_j \times Gl(S)/\mathrm{Spin}\,(n,k) \\
\uparrow \mu & & \uparrow (\mathrm{id}, \mu_0) \\
\pi_F^{-1}(U_j)/SO(n,k) & \xrightarrow{\tilde{\phi}_j} & U_j \times Gl_+(n;\mathbb{R})/SO(n,k),
\end{array} \tag{1}
$$

where $\tilde{\phi}_j$ and $\tilde{\psi}_j$ are the induced trivializations, and such that for each $g \in G$ we have $\phi_j \circ \Phi_g = \Phi_{\tilde{\phi}_j(g)} \circ \psi_j$ on $\mu(g)|_{U_j}$.

4. THERE ARE NO SUCH SPINORS

Let $Gl(S)' = Gl(S)/\pm 1$, $P' = P/\pm 1$, and let $p : P \to P'$, $p_0 : Gl(S) \to Gl(S)'$ be the natural projections. Since ± 1 is a closed normal subgroup of $Gl(S)$, $Gl(S)'$ is a Lie group, $\pi' : P' \to M$ with $\pi' \circ p = \pi$, is a principal bundle with group $Gl(S)'$, and p_0, p are epimorphisms. The group $SO(n,k)$ acts freely on the right upon $Gl(S)'$ by $\rho_Q(p_0(u)) = p_0(u \cdot s)$, where $s \in \Phi^{-1}(Q)$, $Q \in SO(n,k)$. Thus, p_0 induces a map $\tilde{p}_0 : Gl(S)/\mathrm{Spin}\,(n,k) \to Gl(S)'/SO(n,k)$.

Let $A \in Gl_+(n;\mathbb{R})$ and put $\tilde{\Phi}_A : \tilde{p}_0(\mu_0(A \cdot SO(n,k))) \to A \cdot SO(n,k)$ defined by $\tilde{\Phi}_A \circ p_0 = \Phi_A$. There is a unique map $\nu_0 : Gl_+(n;\mathbb{R}) \to Gl(S)'$ such that $(\tilde{\Phi}_A \circ \nu_0)(A) = A$ for every $A \in Gl_+(n;\mathbb{R})$, and since obviously $\tilde{\Phi}_A \circ \rho_Q = \rho_Q \circ \tilde{\Phi}_A$, we have $\rho_Q(\nu_0(A)) = \nu_0(A \cdot Q)$, $Q \in SO(n,k)$.

In the same way we define maps $\nu : F_+(M) \to P'$, $\tilde{p} : P/\mathrm{Spin}\,(n,k) \to P'/SO(n,k)$, and a map $\tilde{\Phi}_g : \tilde{p}(\mu(g)) \to SO(M,g)$ for each $g \in G$.

LEMMA. *ν_0 is a Lie group homomorphism and* $\ker \nu_0 \subset \mathbb{R}^+$. ■

Proof. Let $U \in \{U_j\}_{j\in J}$ and let $\phi : \pi_F^{-1}(U) \to U \times Gl_+(n;\mathbb{R})$, $\psi : \pi^{-1}(U) \to U \times Gl(S)$ be trivializations of $F_+(M)$ and P, respectively, that make the diagram (1) commutative. Let ψ', with $\psi' \circ p = p_0 \circ \psi$, be the induced trivialization of P'. Then the following diagram is commutative

$$
\begin{array}{ccc}
\pi'^{-1}(U) & \xrightarrow{\psi'} & U \times Gl(S)' \\
\uparrow \nu & & \uparrow (\mathrm{id}, \nu_0) \\
\pi_F^{-1}(U) & \xrightarrow{\phi} & U \times Gl_+(n;\mathbb{R}),
\end{array} \tag{2}
$$

and, given ϕ, such a map ψ' is unique, as it is easily checked. Let $A \in Gl_+(n;\mathbb{R})$, so that $\gamma = \lambda_A \circ \phi$ is another trivialization of $F_+(M)$ on $\pi_F^{-1}(U)$. Then there must be some unique element $\tau_0(A) \in Gl(S)'$ such that the trivialization $\lambda_{\tau_0(A)} \circ \psi'$ of P' on $\pi'^{-1}(U)$, together with γ, makes diagram (2) commutative. Thus, if $z \in$ $\in \pi_F^{-1}(U)$,

$$(\lambda_{\tau_0(A)} \circ \psi' \circ \nu)(z) = \tau_0(A) \cdot (\psi'(\nu(z))) = \tau_0(A) \cdot \nu_0(\phi(z)) = \nu_0(A \cdot \phi(z)).$$

If we put $B = \phi(z)$, we conclude that for all $A, B \in Gl_+(n; \mathbb{R})$ we have $\nu_0(A \cdot B) = \tau_0(A)\, \nu_0(B)$, and since $\nu_0(1) = 1$ because $\mu_0(SO(n, k)) = \text{Spin}\,(n, k)$, then $\nu_0(A \cdot B) = \nu_0(A)\, \nu_0(B)$. Also, ν_0 is smooth as a consequence of the following commutative diagram

$$\begin{array}{ccc} Gl_+(n; \mathbb{R}) & \xrightarrow{\nu_0} & Gl(S)' \\ \downarrow & & \downarrow \\ Gl_+(n; \mathbb{R})/SO(n, k) & \xrightarrow{\tilde{p}_0 \circ \mu_0} & Gl(S)'/SO(n, k), \end{array}$$

which stands for a homomorphism of smooth principal bundles where $\tilde{p}_0 \circ \mu_0$ is smooth and $\rho_Q \circ \nu_0 = \nu_0 \circ \rho_Q$ for each $Q \in SO(n, k)$. Therefore ν_0 is a Lie group homomorphism.

Now, $\nu_0 | SO(n, k)$ is injective; since $ZSl(n; \mathbb{R}) \subset SO(n, k) \subset Sl(n; \mathbb{R})$ and $Sl(n; \mathbb{R})/ZSl(n; \mathbb{R})$ is simple, it is clear that $q_0(\ker \nu_0) = 1$, where $q_0 : Gl_+(n; \mathbb{R}) \to Sl(n; \mathbb{R})$ is the projection given by $q_0(A) = (\det A)^{-1/n} A$; thus, $\ker \nu_0 \subset \mathbb{R}^+$, and the Lemma is proved. ■

Let $G = p_0^{-1}(\nu_0(Gl_+(n; \mathbb{R})))$. Then $p_0 : G \to \nu_0(Gl_+(n; \mathbb{R}))$ is a double covering. The transition functions of the trivializations ψ_j take their values in G, whence we can consider henceforth that P is a principal fiber bundle with group G. Since $\nu_0(Gl_+(n; \mathbb{R})) \approx Gl_+(n; \mathbb{R})/\ker \nu_0$, we have that $\Psi = \nu_0^{-1} \circ p : G \to Gl_+(n; \mathbb{R})/\ker \nu_0$ is a double covering. We have also the corresponding epimorphism of principal bundles $\Psi : P \to F_+(M)/\ker \nu_0$. Thus it is clear that the possible new-spinor structures are fixed by the following conditions:

a) the choice of a closed subgroup, $\ker \nu_0$, of $\mathbb{R}^+$;

b) a double covering $\Psi : G \to Gl_+(n; \mathbb{R})/\ker \nu_0$ such that

c) $\Psi^{-1}(SO(n, k)) \approx \text{Spin}\,(n, k)$;

d) the existence of a principal bundle $\pi : P \to M$ with group G, and an epimorphism of principal bundles $P \to F_+(M)/\ker \nu_0$ corresponding to Ψ.

The new notion of spinor should be that of a cross section of a vector bundle associated to P by a representation of G.

Let us consider the different cases, which correspond essentially to the different closed subgroups of $\mathbb{R}^+$. These are 1, $\mathbb{R}^+$ and $Z(r) = \{e^{mr}\}_{m \in \mathbb{Z}}$ $r > 0$. We recall that $\pi_1(Gl_+(n; \mathbb{R}))$ is 0, $\mathbb{Z}$ or $\mathbb{Z}_2$ when n is 1, 2 or > 2, respectively.

We start analysing the general case $n > 2$. Then $\text{Spin}_+(n, k)$ is always connected and $\pi_1(Gl(n;\mathbb{R})) = \pi_1(Sl(n;\mathbb{R})) = \mathbb{Z}_2$. By the remark made in the first paragraph of section 2, it is clear that for $\ker \nu_0 = 1$ or $\mathbb{R}^+$ we have

I. $\ker \nu_0 = 1$, $G = DGl(n;\mathbb{R})$.

II. $\operatorname{Ker} \nu_0 = \mathbb{R}^+$, $G = DSl(n;\mathbb{R})$,

due to condition c).

If $\ker \nu_0 = Z(r)$, then $\mathbb{R}^+/\ker \nu_0 \approx U(1)$, the isomorphism being given by $[x] \to \exp\left(\frac{2\pi i}{r} \log x\right)$, and $Gl_+(n;\mathbb{R})/\ker \nu_0 \approx U(1) \times Sl(n;\mathbb{R})$. Therefore, $\pi_1(Gl_+(n;\mathbb{R})/Z(r)) = \mathbb{Z} \times \mathbb{Z}_2$. We consider the group $DU(1) \times DSl(n;\mathbb{R})$; on each factor we have the involution i that maps a point into the other point on the same fiber. Then (id, id) and (i, i) act on that group as a group of transformations isomorphic to $\mathbb{Z}_2$. Thus, we have three candidates for G, namely $DU(1) \times Sl(n;\mathbb{R})$, $U(1) \times DSl(n;\mathbb{R})$, $(DU(1) \times DSl(n;\mathbb{R}))/\mathbb{Z}_2$. The former makes $\Psi^{-1}(SO(n,k)) = \mathbb{Z}_2 \times SO(n,k)$ and therefore can be dismissed because it contradicts condition c). The other two give correctly a group isomorphic to $\operatorname{Spin}(n,k)$. Therefore we get

III. $\ker \nu_0 = Z(r)$, $G = U(1) \times DSl(n;\mathbb{R})$.

IV. $\ker \nu_0 = Z(r)$, $G = (DU(1) \times DSl(n;\mathbb{R}))/\mathbb{Z}_2$.

Remark. The principal bundles $F_+(M)/\ker \nu_0$ are $F_+(M)$ for case I, $F_+(M)/\mathbb{R}^+$ for case II, and $F_+(M)/Z(r)$ for cases III and IV. The associated tensor fields are the ordinary ones, tensor 0-densities, and tensor densities of imaginary weight ([2], [3]), respectively.

For $n = 1$, and for $n = 2$ $k = 1$, $\operatorname{Spin}_+(n,k)$ is not connected. Therefore the condition c) is irrelevant and we have the possibilities $G = \mathbb{Z}_2 \times Gl_+(n;\mathbb{R})$ if $\ker \nu_0 = 1$, $G = \mathbb{Z}_2 \times Sl(n;\mathbb{R})$ if $\ker \nu_0 = \mathbb{R}^+$, and $G = \mathbb{Z}_2 \times U(1) \times Sl(n;\mathbb{R})$ or $G = DU(1) \times Sl(n;\mathbb{R})$ if $\ker \nu_0 = Z(r)$. The corresponding spinors of the three first cases would be equivalent to non-ordered pairs $(K, -K)$ of tensors, tensor 0-densities or tensor densities of imaginary weight; and for the last one, we would have tensor densities of double imaginary weight. For $n = 1$, there are no other possible types, whereas for $n = 2$ $k = 1$ we would also have the four types of structures listed above.

Now the anticlimax comes in. By the results quoted in section 2, any finite-dimensional representation of G factorizes on Ψ and a finite-dimensional representation of $\Psi(G)$. Thus we arrive to the contradiction that $-1 \in \operatorname{Spin}(n,k)$ must act on S as the identity instead as -1. Therefore, leaving out the rather trivial cases when unordered pairs $(K, -K)$ of tensors or tensor densities are permitted, we conclude that *there are no finite-dimensional true spinors compatible with the family of pseudo-Riemannian metrics of a given index on a smooth oriented manifold.*

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